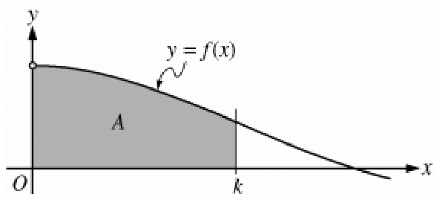
Previous Talk: September 24, 2016

Inspired by #84 BC 2016 International (only 44% answered correctly; originally an area problem)



The figure above shows the region A, which is bounded by the *x*- and *y*- axes, the graph of $f(x) = \frac{sinx}{x}$ for $x > 0$, and the vertical line $x = k$. The region A is revolved around the *x*-axis to form a solid. If *k* increases at a rate of $\frac{3}{\pi}$ units per second, how fast is the volume of the solid increasing when $k = \frac{5\pi}{6}$?

Inspired by #90 BC 2016 International (only 44% answered correctly)

A water vessel has the shape of a right circular cone. The height of the vessel is 20 cm, and the radius of the opening is 5 cm. Water is poured into the vessel at a constant rate of 3 $\frac{cm^3}{sec}$. What is the rate at which the water level is rising when the depth of the water in the cup is 4 cm? (The volume of a cone of height *h* and radius *r* is given by $V = \frac{1}{3}\pi r^2 h$.)

Inspired by #24 AB 2016 International (only 29% answered correctly)

Gravel is deposited into a pile with a circular base. The volume V of the pile is given by $V = \frac{2r^3}{3}$, where r is the radius of the base, in feet. The circumference of the base is increasing at a constant rate of 6π feet per hour. When the circumference of the base is 12π feet, what is the rate of change of the volume of the pile, in cubic feet per hour?

Introduction to Related Rates

Verbal	Leibniz Notation	LaGrange Notation
1. At $t = 3$ minutes, the volume of a sphere is increasing at a rate of 5 $\frac{ft^3}{min}$.		
2.	$\frac{dh}{dt} = -3\ in/sec$ at $t = 10\ sec.$	
3.		$A'(4) = 7\ mm^2/day$
4. The radius of the cylinder is a constant $5\ ft$.		
5.	$\frac{dC}{dt} = 6\ m/day$ at $t = 4\ days.$	
6. The water level in a tank is decreasing at a rate of 4 in/min at $t = 3\ min$.		
7.		$r'(6) = -3\ cm/min$
8.		$S'(1) = \frac{3}{2}\ in^2/hour$
9. The area of a puddle is increasing at a rate of 10 $sq.\ in\ per\ min$ at $t = 6\ min$		
10.	$\frac{dS}{dt} = 12\ cm^2/sec$ at $t = 20\ sec.$	
11.		$T'(30) = -6$ °F$/min$
12.	$\frac{dV}{dt} = -\frac{1}{2}\frac{gal}{min}$ at $t = 3\ min.$	
13.		$w'(4) = 2\ m/week$
14. The length of a diagonal of the rectangle is changing at a rate of 6 $\frac{ft}{hr}$.		
15.	$\frac{dA}{dt} = -4\frac{ft^2}{hr}$ at $t = 2\ hrs.$	

Page Two, **Related Rates**

The next thing that we will have to do is to learn to take derivatives with respect to time. For each of the following formulas, please identify the formula and then take the derivative (with respect to time.) I will do the first three as examples and then you will do the others on your own.

1. $A = \pi r^2$
2. $V = \pi r^2 h$
3. $a^2 + b^2 = c^2$
4. $S = 2\pi rh + 2\pi r^2$
5. $A = s^2$
6. $S = 4\pi r^2$
7. $P = 2l + 2w$
8. $V = \frac{1}{3}\pi r^2 h$
9. $V = s^3$
10. $x^2 + y^2 = 25$
11. $C = 2\pi r$
12. $V = \frac{4}{3}\pi r^3$
13. $l^2 + 6^2 = h^2$
14. $P = 4s$
15. $A = lw$
16. $A = \frac{1}{2} bh$

Building a Bridge Between Implicit Differentiation and Related Rates

1. Let x and y be functions of time t that are related by the equation $x^2y - 3x = 6$. At time $t = 4$, the value of x is 2 and $\frac{dy}{dt}$ is 5. Find the value of $\frac{dx}{dt}$ at time $t = 4$.

2. Let x and y be functions of time t that are related by the equation $2x^2 + 3y^2 - 4xy = 36$. At time $t = 2$, the value of x is 2, the value of y is -2, and the value of $\frac{dy}{dt}$ is -3. Find the value of $\frac{dx}{dt}$ at time $t = 2$.

3. Consider the curve given by $x^{2/3} + y^{2/3} = 5$. At time $t = 1$, the value of x is 1, y is 8 and $\frac{dx}{dt} = -6$. Find the value of $\frac{dy}{dt}$ at time $t = 1$.

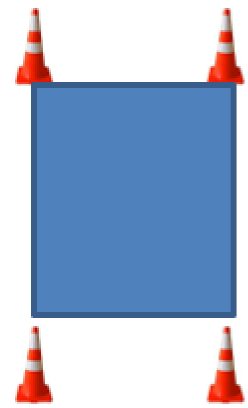
4. Let x and y be functions of time t that are related by the equation $4x^2 + y^2 - 8x + 4y + 4 = 0$. At time $t = 5$, the value of x is 1, the value of y is negative, and the value of $\frac{dy}{dt}$ is 0. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

5. Consider the curve given by $x^2 = 4xy - 6$. At time $t = 2$, the value of x is 3 and $\frac{dx}{dt} = -2$. Find the value of $\frac{dy}{dt}$ at time $t = 2$.

6. Consider the curve given by $y^3 = 4 + 2xy$. At time $t = 3$, the value of y is 2 and $\frac{dy}{dt} = 5$. Find the value of dx/dt at time $t = 3$.

The Angry Aardvarks of Sir Isaac Newton High School are celebrating their victory over their arch-rivals, the Placid Possums of Gottfried Wilhelm Leibniz Middle School with a **related rates extravaganza** /homecoming dance.

1. The teachers have roped off a rectangular dance floor using thick elastic cords. The dance floor was originally 18 ft wide and 31 ft long, BUT so many Aardvarks have shown up to dance and celebrate that they need to enlarge the dance floor. Mr. Julio is in the northeast corner of the floor and begins dragging his cone east at a constant rate of 4 ft/sec. Ms. Romea is in the southwest corner of the dance floor and drags her cone south at a constant rate of 3 ft/sec. A freshman Aardvark in the southeast corner drags her cone diagonally to maintain the rectangular shape of the dance floor.



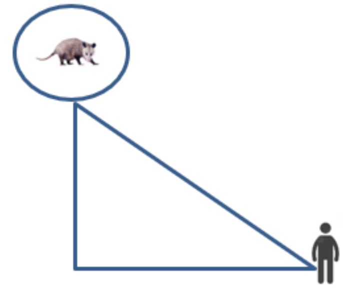
A) After 3 seconds, how fast is the perimeter of the dance floor changing?

B) After 3 seconds, how fast is the area of the dance floor changing?

C) After 3 seconds, how fast is the freshman Aardvark moving in the diagonal direction?

D) Ms. Romea notices that the elastic cord maintains its volume as she is stretching it in the southward direction. At 3 seconds, she notices the cord has a diameter of 4 inches (1/3 foot). How fast is the radius of the elastic cord changing at this time?

2. As part of their celebration of the tennis team victory, the Angry Aardvarks plan to pummel a Placid Possum pinata. The Possum pinata is being hoisted vertically at a constant rate of 2 ft/sec. Ms. Romea is standing 8 ft away from a spot directly beneath the pinata.



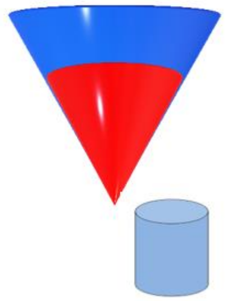
A) When the pinata reaches a height of 6 ft, how fast is the distance between Ms. Romea and the pinata changing at that time?

B) Ms. Romea rotates her head to maintain her line of sight on the pinata as it rises. How fast is the angle of her line of sight changing when the pinata reaches a height of 6 ft.?

C) A spotlight that is directly above Ms. Romea's head and 12 feet above the floor casts a shadow of the pinata on the floor. As the pinata continues to rise at a constant rate of 2 ft/sec and reaches the height of 6 feet above the floor, how fast is the shadow of the pinata moving across the gym floor?

D) When the pinata is busted open, candy falls to the floor in a conical pile. Ms. Romea notices that the diameter of the pile is always 3 times the height. If the height of the pile increases at a rate of 4 inches/sec when it reaches a height of 5 inches, then how fast is the volume of candy in the pile changing at that time? (Volume of a cone is $V = \frac{1}{3}\pi r^2 h.$)

3. Meanwhile, Mr. Julio decides to get some punch from the refreshment table. The punch is being served from a conical punchbowl with a spigot at the bottom.



A) Mr. Julio notes that the punchbowl has a diameter of 18 inches and a height of 12 inches. As he serves himself punch from the spigot, he notes that the height of the punch in the bowl is decreasing at a rate of ½ inch per second and the height of the punch at this moment is 6 inches. How fast is the volume of punch in the bowl decreasing at this moment?

B) How fast is the circumference of the surface of the punch decreasing at this moment?

C) Mr. Julio is serving the punch into a cylindrical paper cup with a radius of 4 inches and a height of 5 inches. How fast is the height of the punch in his cup increasing at this moment?

Can you continue the saga of the Angry Aardvarks? Put someone up on a ladder and have a possum pull the bottom out. Have someone blowing up a spherical balloon. There are infinite possibilities here! Send your related rates problem to dixross@austin.rr.com

The Mighty Pflugerville Panthers

vs. the Stoney Point Tigers
in a Related Rates Extravaganza

vs.

1. The Patriotic Panther is watching as the American flag rises up the flagpole at a rate of 6 feet per minute. If the Panther is standing 40 feet away from the base of the flagpole, how fast is the distance between the Panther and the flag changing when the flag is 30 feet from the ground?

2. As the players run onto the field, the incredibly observant Panther notices that the cheerleaders have a new rectangular run-through sign that appears to be made of rubber. It maintains a constant area of 200 square feet, but its length is increasing at a rate of ½ foot per minute. How fast is the width changing when the length is 25 feet?

3. After the first half of play, the high-scoring Panther goes to get some Gatorade from the cylindrical cooler which has a radius of 10 inches and a height of 24 inches. If the height of Gatorade in the cooler is decreasing at a rate of 2 inches per minute, how fast is the volume of Gatorade in the cooler changing? ($V = \pi r^2 h$)

4. The Gatorade is flowing into a conical cup at rate of 10 cubic inches per minute. If the cup has a radius of 3 inches and a height of 4 inches, then how fast is the height of the Gatorade in the cup increasing when it reaches a height of 2 inches? ($V = \frac{1}{3}\pi r^2 h$)

5. The Stoney Point Tigers, in a last-ditch desperate attempt to get on the scoreboard, attempt an on-sides kick. The amused Panther standing 40 feet away watches as the ball travels straight up at a rate of 8 feet per second. How fast is the angle of his line of sight changing when the ball is 30 feet in the air?

6. A 55-foot light post that is 40 feet away casts the shadow of the ball on the ground as it travels straight downward at a rate of 10 feet per second. How fast is the shadow of the ball traveling on the ground as it conks the incompetent 5-foot Stoney Point Tiger in the head?

Go Panthers! Beat the Tigers, who are worthy opponents and beating them will bring honor and glory to our school.

After Unit 5, we re-visit our Angry Aardvarks who continue their domination of the Peculiar Possums of Gottfried Wilhelm Leibniz Middle School. They are inflating a giant spherical hot air balloon to fly over their rival's school and drop water balloons on the un-suspecting Possums.

Variation on 2007#5 (No Calculator)

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 36 feet when $t = 7$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) How do you know that $r'(t)$ is decreasing for entire interval $0 \le t \le 12$?

(b) Estimate the value of $r''(6)$ using the data in the table. Explain the meaning of the value of $r''(6)$ in the context of the problem.

(c) Estimate the radius of the balloon when $t = 7.2$ using the tangent line approximation at $t = 7$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(d) Is there guaranteed to be a time in the interval $0 \le t \le 12$ such that $r'(t) = 1.8$ feet per minute? Justify your answer.

(e) Is there guaranteed to be a time such that $r''(t) = -0.35$ ft/min/min? Justify your answer.

(f) Find the rate of change of the volume of the balloon with respect to time when $t = 7$. Indicate units of measure.

<u>After Unit 6</u>, we re-visit our Angry Aardvarks. Our much put-upon Possums have had enough! They have commandeered the hot air balloon and are now flying over Sir Isaac Newton High School and dropping flaming bags of Possum poo on the un-suspecting and now greatly Aggrieved Aardvarks.

Variation on 2007#5 (No Calculator)

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function *r* of time *t*, where *t* is measured in minutes. For $0 < t < 12$, the graph of *r* is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 12 feet when $t = 2$. (Note: The surface area of a sphere of radius *r* is given by $S = 4\pi r^2$.)

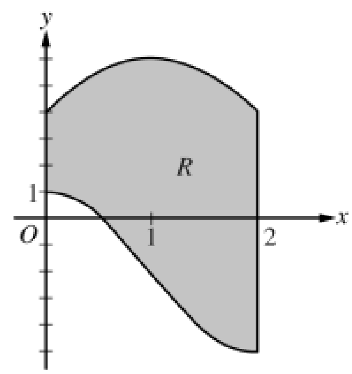
(a) Find the rate of change of the surface area of the balloon with respect to time when $t = 2$. Indicate units of measure.

(b) Use a left Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\, dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\, dt$ in terms of the radius of the balloon.

(c) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\, dt$? Give a reason for your answer.

After Unit 8

2019 AP® CALCULUS AB FREE-RESPONSE QUESTIONS



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x - 1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

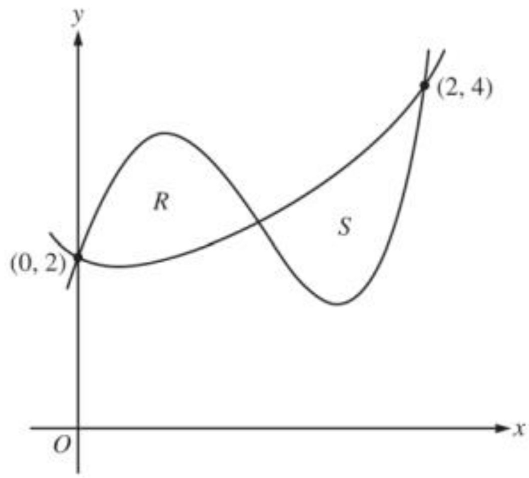
   (a) Find the area of R.

   (b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \dfrac{1}{x + 3}$. Find the volume of the solid.

   (c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

NEW PART C) A vertical line in the xy −plane travels from left to right along the base of the solid described in part (b). The vertical line is moving at a constant rate of 4 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = 1$.

Make part (c) into part (d).



2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2-2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

   (a) Find the sum of the areas of regions R and S.

   (b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

   (c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

NEW PART C) A vertical line in the xy −plane travels from left to right along the base of the solid described in part (b). The vertical line is moving at a constant rate of 3 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = 1.2$.

Make part (c) into part (d).

AFTER YOU HAVE RE-VISITED MOTION:

AP® Calculus AB 2022 Free-Response Questions

6. Particle P moves along the x-axis such that, for time $t > 0$, its position is given by $x_P(t) = 6 - 4e^{-t}$.

   Particle Q moves along the y-axis such that, for time $t > 0$, its velocity is given by $v_Q(t) = \frac{1}{t^2}$. At time $t = 1$, the position of particle Q is $y_Q(1) = 2$.

   (a) Find $v_P(t)$, the velocity of particle P at time t.

   (b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

   (c) Find $y_Q(t)$, the position of particle Q at time t.

   (d) As $t \to \infty$, which particle will eventually be farther from the origin? Give a reason for your answer.

(e) NEW PART: How fast is the distance between particles P and Q changing at $t = 2$?

2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) Evaluate $\int_0^{3.5} v(t)\,dt$, and evaluate $\int_0^{3.5} |v(t)|\,dt$. Interpret the meaning of each integral in the context of the problem.

(d) A second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time t are the two particles moving with the same velocity?

Edit of part (d) A second particle moves along the y-axis with position given by $y_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time $t$are the two particles moving with the same velocity.

New part (e) At time $t = 3$, determine how fast the distance between the two particles is changing.

CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

New part (e): The tank into which water is being pumped and removed is in the shape of a right circular cylinder with a radius of 10 feet and a height of 8 feet. If a liter of water is approximately 61 cubic inches, then about how fast in inches/hour is the water level in the tank changing when $t = 3$ hours. Hint: 1 foot = 12 inches.

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

   (a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

   (b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

   (c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

New part (d): Let x and y be functions of time t that are related by the equation $y^3 - xy = 2$. At time $t = 4$, the value of x is -1, the value of y is 1, and the value of $\frac{dx}{dt}$ is -2. Find the value of $\frac{dy}{dt}$ at time $t = 4$.