Particle Motion

Background and Definitions

- **Vector**: A quantity that has both magnitude and direction.

  Examples: Displacement, velocity, force.

  Graphically represented by an arrow, or a directed line segment.

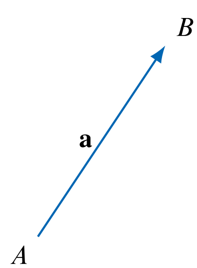
  Length of the arrow: magnitude; Arrow points in the direction of the vector.

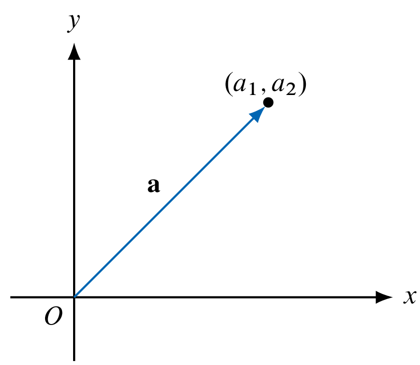
  Notation: $\mathbf{a}$, $\vec{a}$

- Suppose a particle moves along a line segment from A to B.

  The **displacement vector a** has **initial point** A and **terminal point** B.

  Notation: $\mathbf{a} = \overrightarrow{AB}$





- Often we use a coordinate system and treat vectors algebraically.

  Initial point: origin, Terminal point of **a**: (a_1, a_2) (**components** of **a**)

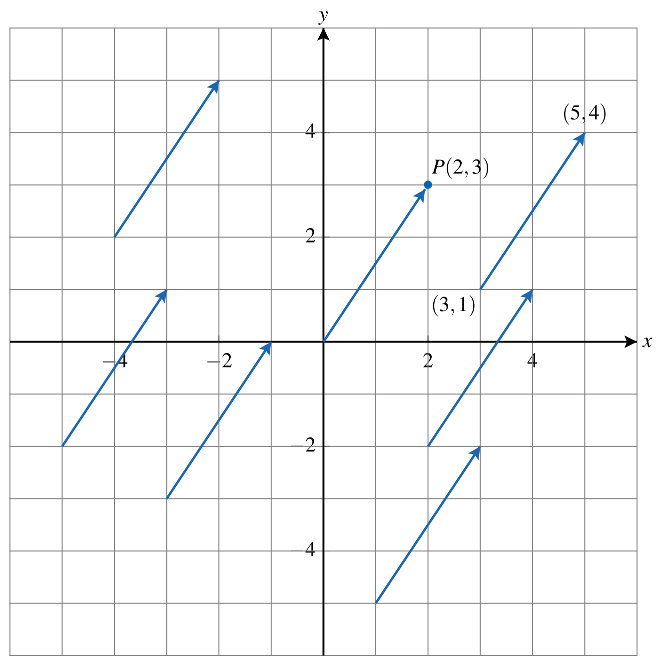
  Notation:

  $\mathbf{a} = \langle a_1, a_2 \rangle$: the ordered pair that refers to a vector, or directed line segment.

  (a_1, a_2) : a point in the plane.

• Example:

All of the vectors shown are equivalent to the vector $\overrightarrow{OP} = \langle 2, 3 \rangle$; terminal point $P(2, 3)$.



The terminal point is obtained from the initial point by a displacement of 2 units to the right, and 3 units upward.

All geometric representations of the vector $\mathbf{a} = \langle 2, 3 \rangle$.

The representation $\overrightarrow{OP}$: **position vector** of the point P.

• **Magnitude** or **length** of the vector $\mathbf{a}$:

The length of any of its representations.

Notation: $|\,\mathbf{a}\,|$, $\quad ||\,\mathbf{a}\,||$

$$|\,\mathbf{a}\,| = \sqrt{a_1^2 + a_2^2}$$

- Function: a rule that assigns to eah element in the domain a unique element in the range.

  Vector-valued function or **vector function**:

  A function whose domain is the set of real numbers and whose range is a set of vectors.

  Consider vector functions $\mathbf{r}$ whose values are two-dimensional vectors.

  For every number t in the domain of $\mathbf{r}$, there is a unique vector denoted by $\mathbf{r}(t)$.

  Suppose $f(t)$ and $g(t)$ are the components of the vector $\mathbf{r}(t)$.

  Then f and g are real-valued functions: **component functions** of $\mathbf{r}$.

  Notation: $\mathbf{r}(t) = \langle f(t), g(t) \rangle$

- The connection between vector functions and particle motion:

  Suppose a particle moves in the plane so that at time t the position of the particle is given by the parametric equations $x = f(t)$, and $y = g(t)$.

  Consider the vector function $\mathbf{r} = \langle f(t), g(t) \rangle$.

  $\mathbf{r}(t)$: the position vector of the point $P(f(t), g(t))$.

  Therefore: any continuous vector function $\mathbf{r}$ describes the position of a particle in two dimensions and, also defines a curve C traced out by the tip of the moving vector $\mathbf{r}(t)$.

- The derivative $\mathbf{r}'$ of a vector function:

  $\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$

- Suppose a particle moves along a curve C: position vector at time t is $\mathbf{r}(t)$.

  Velocity vector : $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

  Speed of the particle at time t; the magnitude of the velocity vector:

  $$\text{speed} = |\,\mathbf{v}(t)\,| = |\,\mathbf{r}'(t)\,| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

  Total distance traveled by the particle from time $t = \alpha$ to time $t = \beta$:

  $$\text{distance traveled} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

  Acceleration of the particle at time t: derivative of the velocity:

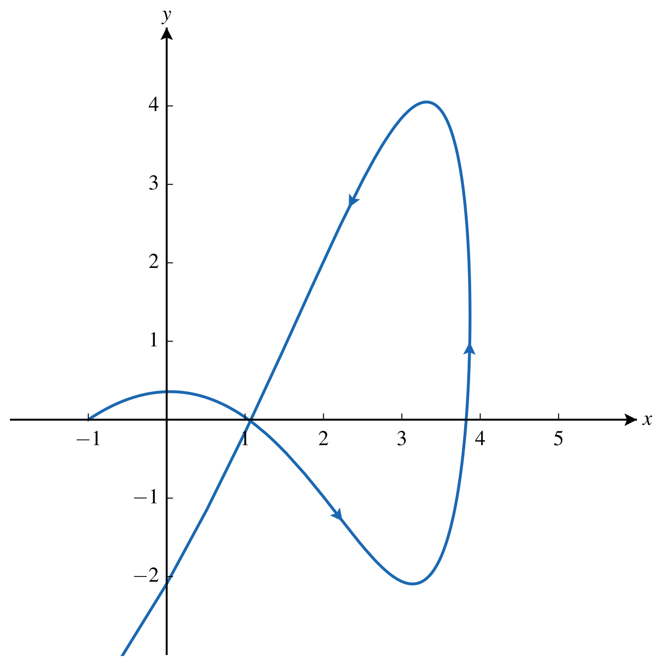
  $$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

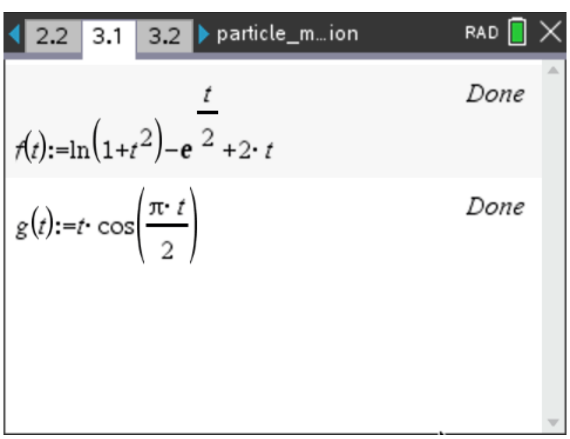
Example 1 Motion in the Plane

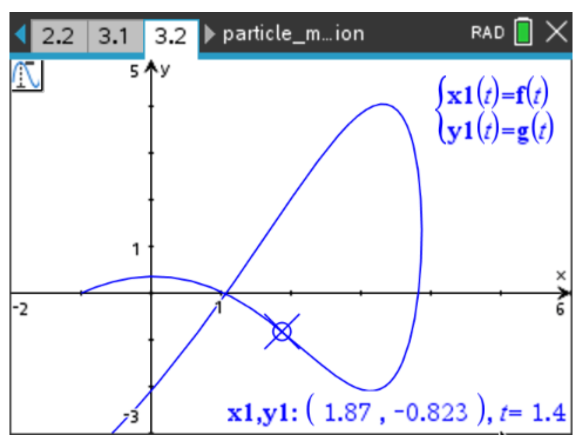
A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, for $t \geq 0$, where

$$x(t) = \ln(1 + t^2) - e^{t/2} + 2t \qquad y(t) = t\cos\left(\frac{\pi t}{2}\right)$$

(a) Sketch the path of the particle. Use arrows to indicate the direction of the particle at t increases.







(b) Find the velocity vector and the acceleration vector at time $t = 1$.

$$x'(t) = \frac{2t}{1+t^2} - \frac{1}{2}e^{t/2} + 2$$

$$x'(1) = \frac{2}{2} - \frac{1}{2}e^{1/2} + 2 = 3 - \frac{1}{2}\sqrt{e}$$

$$y'(t) = 1 \cdot \cos\left(\frac{\pi t}{2}\right) + t \cdot \frac{\pi}{2}(-\sin\left(\frac{\pi t}{2}\right) = \cos\left(\frac{\pi t}{2}\right) - \frac{\pi t}{2}\sin\left(\frac{\pi t}{2}\right)$$

$$y'(1) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2}\cdot 1 = -\frac{\pi}{2}$$

$$\mathbf{v}(1) = \left\langle 3 - \frac{\sqrt{e}}{2}, -\frac{\pi}{2} \right\rangle$$

$$x''(t) = \frac{(1+t^2)\cdot 2 - 2t \cdot 2t}{(1+t^2)^2} - \frac{1}{4}e^{t/2} = \frac{2-2t^2}{(1+t^2)^2} - \frac{e^{t/2}}{4}$$

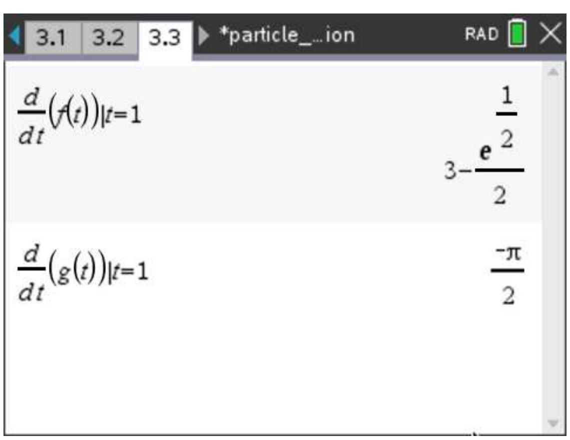
$$x''(1) = 0 - \frac{e^{1/2}}{4} = -\frac{\sqrt{e}}{4}$$

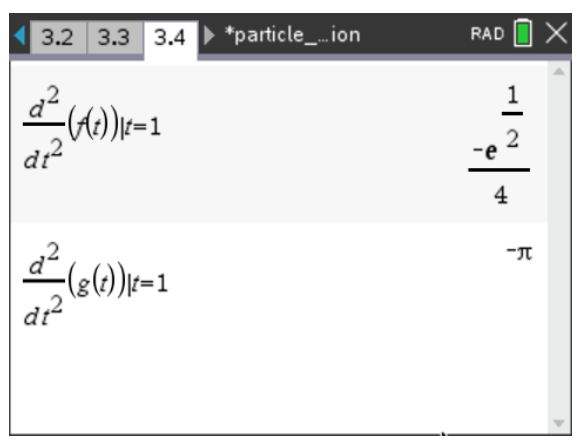
$$y''(t) = -\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right) - \left[\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right) + \frac{\pi t}{2}\cdot\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\right]$$

$$= -\pi\sin\left(\frac{\pi t}{2}\right) - \frac{\pi^2 t}{4}\cos\left(\frac{\pi t}{2}\right)$$

$$y''(1) = -\pi \cdot 1 - \frac{\pi^2}{4}\cdot 0 = -\pi$$

$$\mathbf{a} = \left\langle -\frac{\sqrt{e}}{4}, -\pi \right\rangle$$



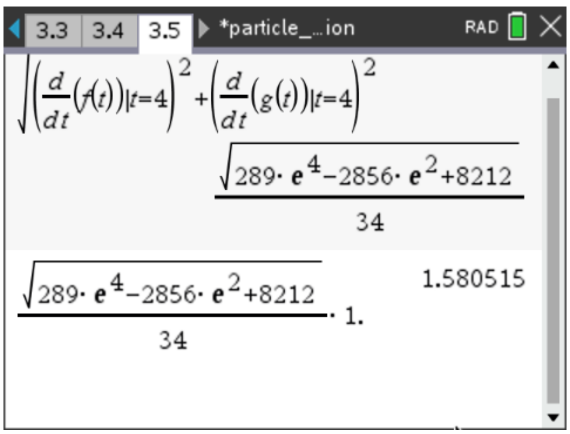


(c) Find the speed of the particle at time $t = 4$.

$$\text{speed} = \sqrt{x'(4)^2 + y'(4)^2}$$

$$= \sqrt{\left(\frac{8}{17} - \frac{1}{2}e^2 + 2\right)^2 + (\cos 2\pi - 2\pi \sin 2\pi)^2}$$

$$= \cdots = \sqrt{1 + \left(\frac{42}{17} - \frac{e^2}{2}\right)^2} = 1.581$$



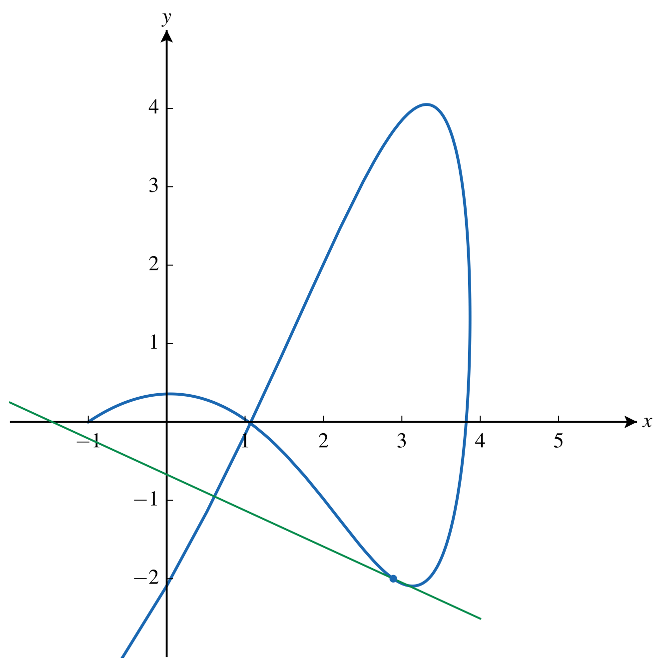
(d) Find an equation of the tangent line to the path of the particle at time $t = 2$.

$$(x(2), y(2)) = (\ln 5 - e + 4, 2 \cdot (-1)) = (\ln 5 - e + 4, -2)$$

$$\frac{dy}{dx} = \frac{y'(2)}{x'(2)} = \frac{-1 - \pi \cdot 0}{\frac{4}{5} - \frac{1}{2}e + 2} = \cdots = -\frac{10}{28 - 5e}$$

An equation of the tangent line:

$$y = -\frac{10}{28 - 5e}(x - (\ln 5 - e + 4)) - 2 = -0.694(x - 2.891) - 2$$



Example 2 Particle Motion and Technology

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, for $t \geq 0$, where

$$\frac{dx}{dt} = \frac{\sin 2t}{1+t^2} \qquad \frac{dy}{dt} = te^{\tan^{-1} t}$$

(a) Find the speed of the particle at time $t = 1$. Find the average speed of the particle over the interval $[1, 5]$.

(b) Find the distance traveled by the particle over the time interval $1 \leq t \leq 2$.

(c) Find the acceleration vector at time $t = 1$.

Solution

(a) Find the speed of the particle at time $t = 1$. Find the average speed of the particle over the interval $[1, 5]$.

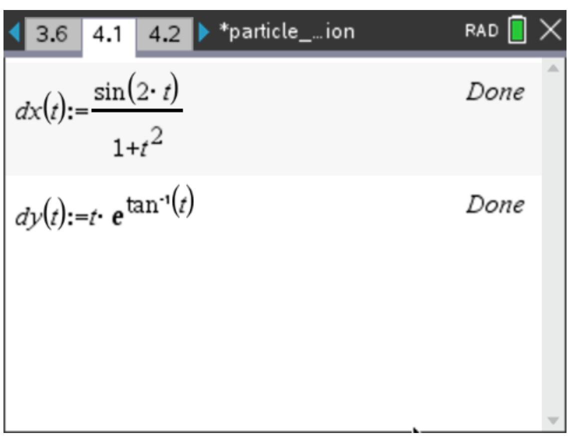
$$\text{speed} = \sqrt{\left(\frac{\sin 2}{2}\right)^2 + \left(e^{\pi/4}\right)^2}$$

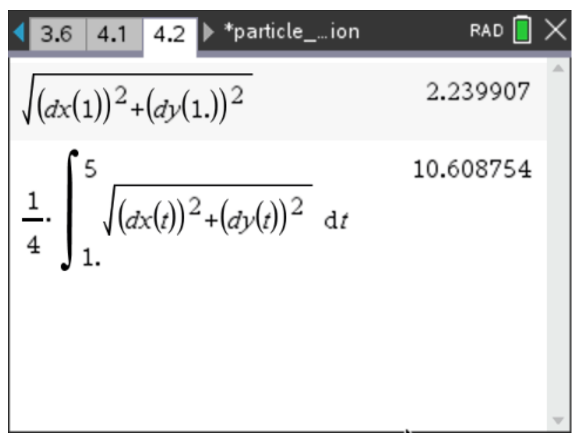
$$= \sqrt{\frac{\sin^2 2}{4} + e^{\pi/2}} = 2.240$$

$$\text{ave speed} = \frac{1}{5-1}\int_1^5 |\mathbf{v}(t)|\,dt = \frac{1}{4}\int_1^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \frac{1}{4}\int_1^5 \sqrt{\left(\frac{\sin 2t}{1+t^2}\right)^2 + \left(t e^{\tan^{-1} t}\right)^2}\,dt$$

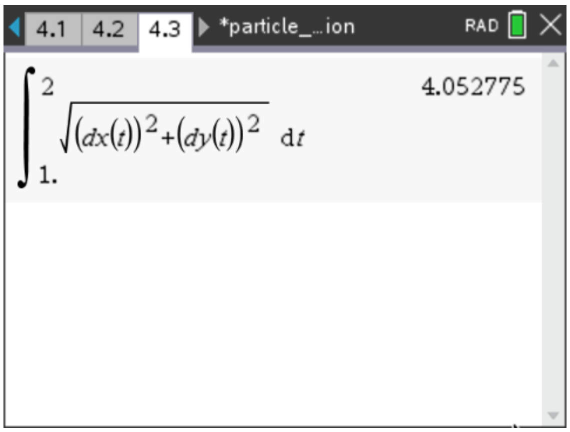
$$= 10.609$$





(b) Find the distance traveled by the particle over the time interval $1 \le t \le 2$.

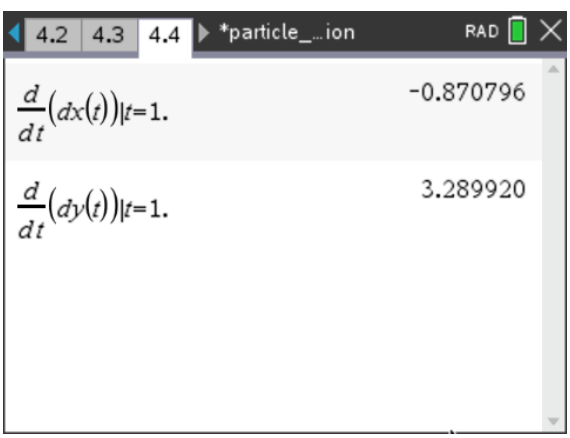
$$\text{distance traveled} = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 4.053$$



(c) Find the acceleration vector at time $t = 1$.

$x''(1) = -0.871 \qquad y''(1) = 3.290$

$\mathbf{a}(1) = \langle -0.871, 3.290 \rangle$



Example 3 New Position

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \ln(t+2) \qquad\qquad \frac{dy}{dt} = \cos(e^{-t^2})$$

for $t \geq 0$. At time $t = 1$ the particle is at position $(3, -4)$.

(a) Find the slope of the tangent line to the curve at position $(3, -4)$.

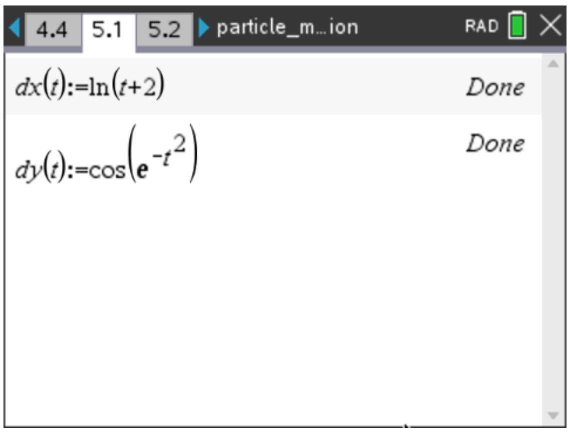
(b) Find the x-coordinate of the position of the particle at time $t = 4$.

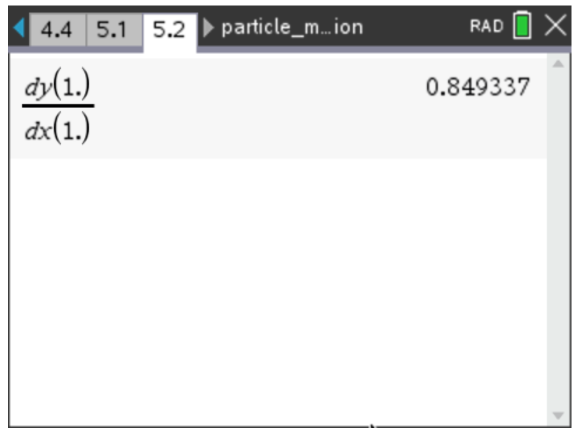
Solution

(a) Find the slope of the tangent line to the curve at position (3, −4).

Find $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at time $t = 1$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{\cos(e^{-1})}{\ln 3} = 0.849$$





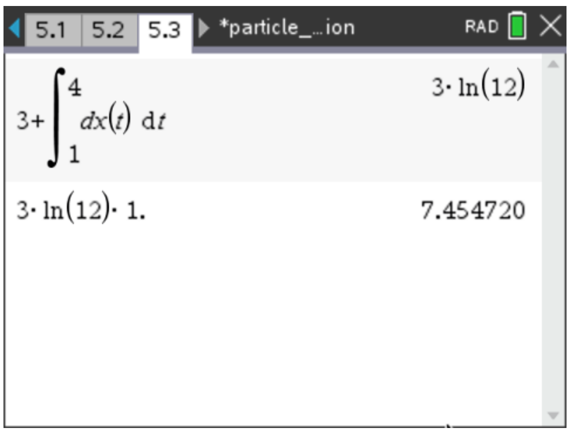
(b) Find the x-coordinate of the position of the particle at time $t = 4$.

$$x(4) = x(1) + \int_1^4 x'(t)\,dt = 3 + \int_1^4 \ln(t+2)\,dt$$

$$= 3 + \Big[(t+2)\ln(t+2) - t\Big]_1^4$$

$$= 3 + [(6\ln 6 - 4) - (3\ln 3 - 1)]$$

$$= 6\ln 6 - 3\ln 3 = 6\ln 2 + 3\ln 3 = \ln 1728 = 3\ln 12 = 7.455$$



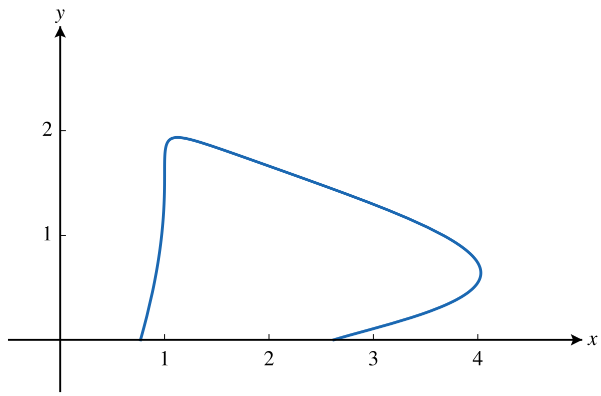
Challenge: Find the y-coordinate of the position of the particle at time $t = 4$.

Example 4 Homework Problem

A particle moving along a curve C in the xy-plane is at position $(x(t), y(t))$ for $0 \le t \le 10$, where

$$y(t) = 0.01(t^3 - 22t^2 + 120t) \qquad\qquad \frac{dx}{dt} = \sin\left(\frac{\pi(t-2)^2}{36}\right)$$

and $x(t)$ is not explicitly given. At time $t = 2$ the x-coordinate of the position of the particle is $x = 1$, and the curve C is shown in the figure.



(a) Find the position of the particle at time $t = 4$.

(b) Find the acceleration vector when the speed is first equal to 1.

(c) Find all times t, $0 \le t \le 10$, for which the tangent line to the curve C is vertical.

(d) Does the particle ever exceed a height of 2? Justify your answer.

(e) Find the average speed of the particle for $0 \le t \le 10$.

Part A (BC): Graphing calculator required
Question 2 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

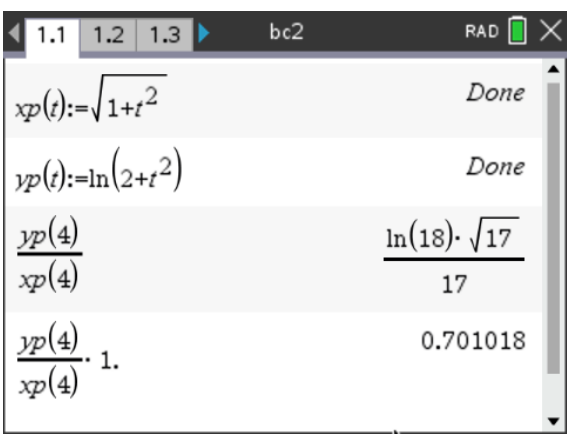
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln(2 + t^2)$. At time $t = 4$, the particle is at the point $(1, 5)$.

	Model Solution	Scoring	
(a)	Find the slope of the line tangent to the path of the particle at time $t = 4$.		
	$\left.\frac{dy}{dx}\right\vert_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$ The slope of the line tangent to the path of the particle at time $t = 4$ is 0.701.	Answer	**1 point**

Solution

$$\left.\frac{dy}{dx}\right|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln(2+4^2)}{\sqrt{1+4^2}} = \frac{\ln 18}{\sqrt{17}} = 0.701$$



Scoring notes:

- To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point: $\frac{y'(4)}{x'(4)} = 0.701$, $\frac{\ln\left(2 + 4^2\right)}{\sqrt{1 + 4^2}}$, or $\frac{\ln 18}{\sqrt{17}}$.

- Note: A response with an incorrect equation of the form "function $=$ constant", such as $\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}$, will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.

Common Errors

(1) Most students recognized the need to find $\frac{y'(4)}{x'(4)}$.

(2) There were some communication errors, especially in equating a function to a specific value.

(3) Some students presented the reciprocal of the correct answer.

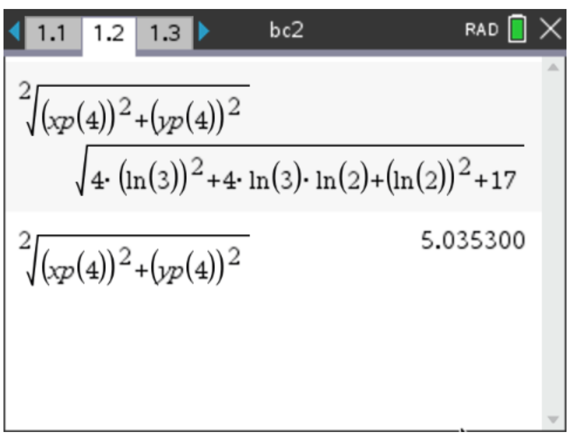
(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

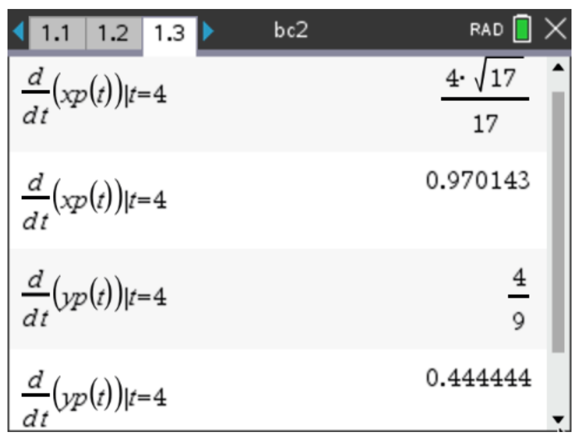
$\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$ The speed of the particle at time $t = 4$ is 5.035.	Speed	**1 point**
$a(4) = \langle x''(4), y''(4)\rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444\rangle$	First component of acceleration	**1 point**
The acceleration vector of the particle at time $t = 4$ is $\langle 0.970, 0.444\rangle$.	Second component of acceleration	**1 point**

Solution

$$\text{Speed} = \sqrt{[x'(4)]^2 + [y'(4)]^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$$

$$\mathbf{a}(4) = \langle x''(4), y''(4)\rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444\rangle$$





Scoring notes:

- To earn any of these points, the setup used to perform the calculation must be evident in the response. The following examples earn the first point: $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$ or $\sqrt{17 + (\ln 18)^2}$ and $\langle x''(4),\, y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$ would earn both the second and third points.

There must be supporting work. (See last item.)

- The second and third points can be earned independently.

The first and the second component of the acceleration vector can be earned independently.

- If the acceleration vector is not presented as an ordered pair, the x- and y-components must be labeled.

Good communication skills are necessary.

- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.

$a(4) = \langle y''(4), x''(4) \rangle = \langle 0.444, 0.970 \rangle$? - 0

- A response which correctly calculates expressions for both $x''(t) = \frac{t}{\sqrt{1+t^2}}$ and $y''(t) = \frac{2t}{2+t^2}$, but which fails to evaluate both of these expressions at $t = 4$, earns only 1 of the last 2 points.

- An unsupported acceleration vector earns only 1 of the last 2 points.

$a(4) = \langle 0.970, 0.444 \rangle$

Common Errors

(1) Equating the speed function with the value of the function at $t = 4$.

(2) Use of the notation $|v|$ without any indication about how the quantity v was defined.

(3) Parentheses errors that resulted in ambiguous or incorrect expressions.

(4) Many students found the components of the acceleration vector symbolically.
In this case: chain rule errors, power rule errors.

(5) Some responses found the length of the acceleration vector.

(6) Incorrect labels or no labels on vector components.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

$y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right)\, dt$	Integrand	**1 point**
	Uses $y(4)$	**1 point**
$= 5 + 6.570517 = 11.570517$ The y-coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570).	Answer	**1 point**

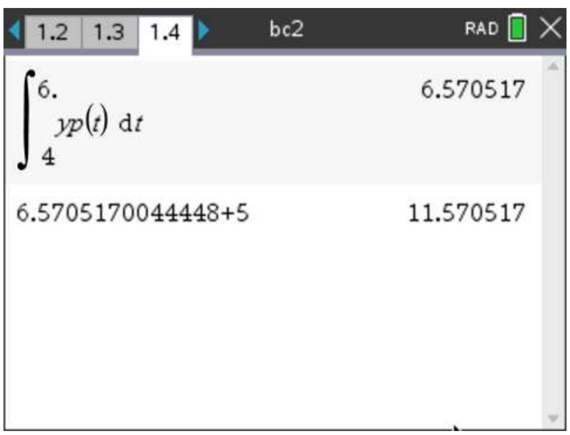
Solution

Given: $y(4) = 5$, Need $y(6)$

$$\int_4^6 y'(t)\, dt = y(6) - y(4) \quad \Rightarrow \quad y(6) = y(4) + \int_4^6 y'(t)\, dt$$

$$y(6) = 5 + 6.570517 = 11.570517$$

The y-coordinate of the particle's position at time $t = 6$ is 11.571.



Scoring notes:

- For the first point, an integrand of $\ln\left(2+t^2\right)$ can appear in either an indefinite integral or an incorrect definite integral.

We want to see the correct integrand.

- A definite integral with incorrect limits is not eligible for the answer point.

A definite integral with correct integrand and incorrect limits cannot resolve to the correct answer. Therefore, this cannot earn the answer point.

- Similarly, an indefinite integral is not eligible for the answer point.

If there are no bounds on the integral, then the response is not eligible for the third point.

- For the second point, the value for $y(4)$ must be added to a definite integral. A response that reports the correct x-coordinate of the particle's position at time $t=6$ as $x(6)=x(4)+\int_4^6 \sqrt{1+t^2}\,dt=11.200$ (or 11.201) instead of the y-coordinate, earns 2 out of the 3 points.

- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).

$\int_4^6 y'(t)\,dt = 6.571$ 1 - 0 - 1

- If the differential is missing:
  - $y(6)=\int_4^6 \ln\left(2+t^2\right)$ earns the first point and is eligible for the third.
  - $y(6)=\int_4^6 \ln\left(2+t^2\right)+y(4)$ does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
  - $y(6)=y(4)+\int_4^6 \ln\left(2+t^2\right)$ earns the first two points and is eligible for the third.

Common Errors

(1) Presentation of only an indefinite integral.

(2) Failure to add the initial position $y(4)$.

(3) Attempts to find $\int \ln(2+t^2)\,dt$ using integration by parts.

Consider $I = \int \ln(2+t^2)\,dt$

$$u = \ln(2+t^2) \qquad dv = dt$$

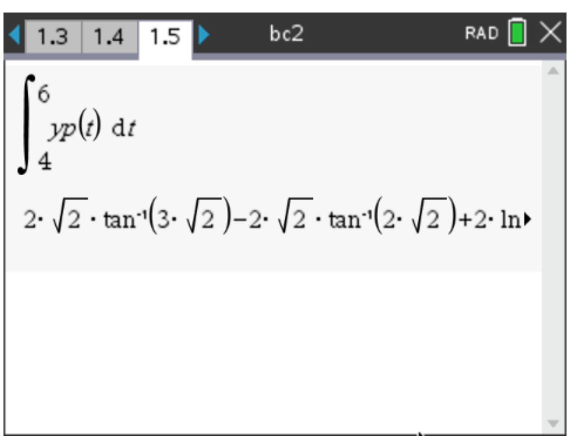
$$du = \frac{2t}{2+t^2}\,dt \qquad v = t$$

$$I = t\ln(2+t^2) - \int \frac{2t^2}{2+t^2}\,dt = t\ln(2+t^2) - \int \left(2 - \frac{4}{2+t^2}\right)\,dt$$

$$= t\ln(2+t^2) - 2t + 4\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)$$

$$= t\ln(2+t^2) - 2t + 2\sqrt{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)$$

(4) CAS results for $\int_4^6 \ln(2+t^2)\,dt$

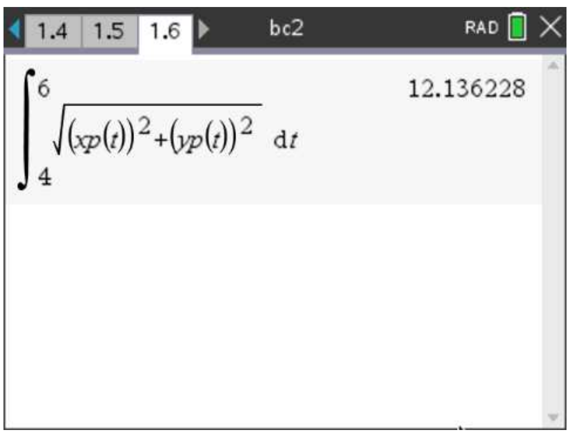


(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

$\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	Integrand	**1 point**
$= 12.136228$ The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136.	Answer	**1 point**

Solution

$$\text{Distance} = \int_4^6 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = 12.136$$



Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.

$$\int_{\square}^{\square} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

- To earn the second point, a response must have earned the first point and must present the value 12.136.

- An unsupported answer of 12.136 does not earn either point.

Common Errors

(1) Presentation errors: missing parentheses.

(2) Presentation of the definite integral $\int_4^6 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dt$.

(3) Presentation of the definite integral: $\int_4^6 \frac{dy}{dx}\, dt$