Particle Motion

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(6) Calculus with Parametric Curves: Particle Motion

Background

Let $s(t)$ be the position function of a particle moving along a line.

$\frac{\Delta s}{\Delta t}$: average velocity of the particle over a time interval of length Δt

$v = \frac{ds}{dt}$: instantaneous velocity of the particle, rate of change of position with respect to time.

$a(t) = v'(t) = s''(t)$: acceleration; instantaneous rate of change of velocity with respect to time.

Example 1 Particle Motion, All Things Considered

A particle moves along a straight line. For $t \geq 0$, the position of the particle is given by $s(t) = 2t^3 - 21t^2 + 60t$, where t is measure in seconds and s in meters. At time $t = 0$, the particle is at position $s = 0$.

(a) Find the velocity at any time t.

The velocity function is the derivative of the position function.

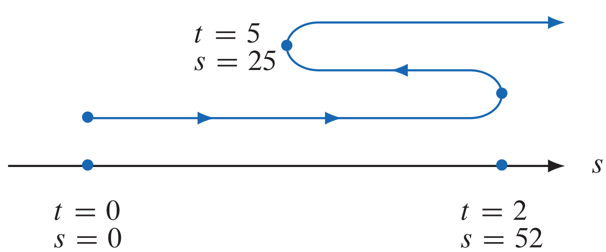
(b) Find the velocity at time $t = 3$ and $t = 6$.

(c) Find all times t when the particle is at rest.

(d) When is the particle moving in the positive direction (to the right)?

The particle is moving in the positive direction when $v(t) > 0$.

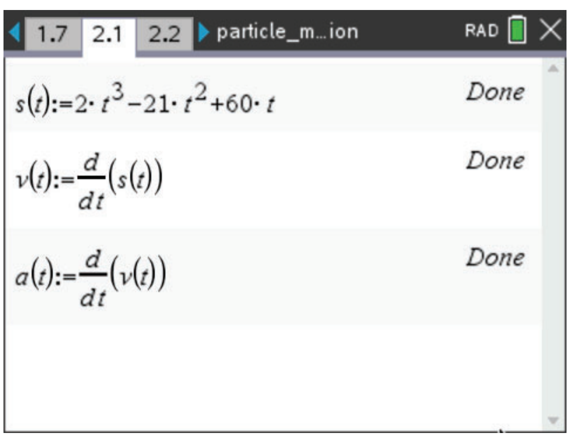
(e) Draw a diagram to represent the motion of the particle.

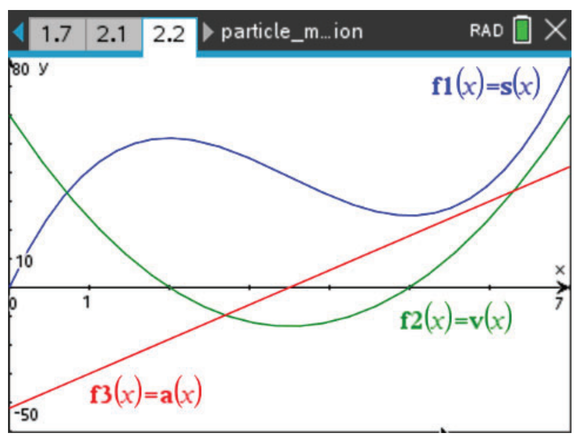


(f) Find the total distance traveled by the particle over the time interval [0, 4].

(g) Find the acceleration of the particle at time t and at time $t = 6$ seconds.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 7$.

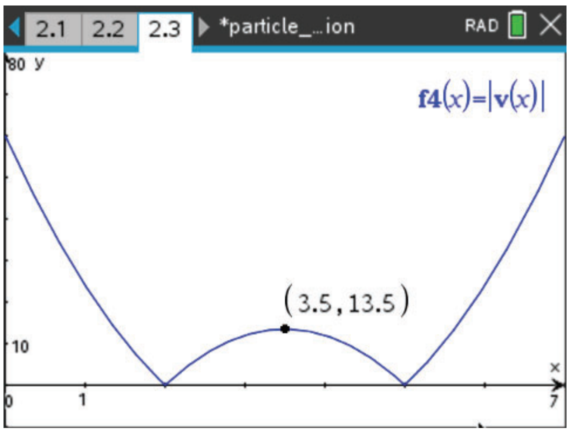




(i) When is the particle speeding up? When is it slowing down?

The speed of the particle is the absolute value of the velocity.

$|v(t)|$ is the speed at time t.

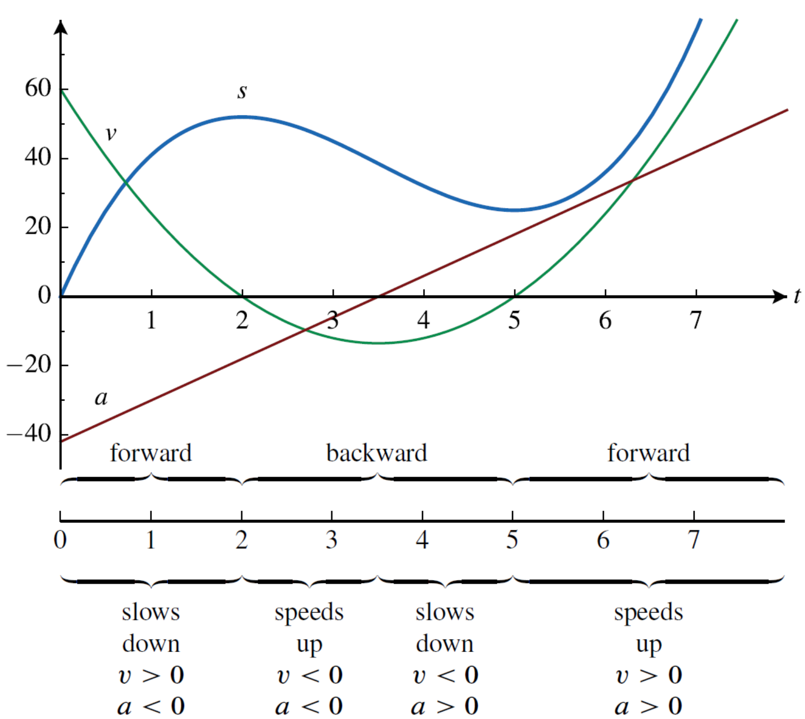


Slowing down: $0 \le t < 2,\ 3.5 < t < 5$

Speeding up: $2 < t < 3.5,\ t > 5$

Alternate method to find these intervals?

Here is a summary of the motion of the particle:



Particle Motion Lingo

(1) Expression:

The initial position, or velocity, or acceleraton, of the particle is . . .

Mathematical Translation:

The word *initial* usually means at time $t = 0$. In many particle motion problems, we are often given the initial positon $s(0)$, or the initial velocity $v(0)$, or the initial acceleration $a(0)$, of the particle.

(2) Expression:

The particle is at position $s = s_1$ at time $t = t_1$.

Mathematical Translation:

This expression conveys the position of the particle at a spcific time: $s(t_1) = s_1$.

(3) Expression:

The particle is at rest.

Mathematical Translation:

This expression means that the velocity is 0, that is, $v(t) = 0$.

(4) Expression:

The particle is moving to the right.

Mathematical Translation:

This expression tells us something about the velocity of the particle. The velocity of the particle is greater than 0, $v(t) > 0$.

(5) Expression:

The particle is moving to the left.

Mathematical Translation:

This expression tells us something about the velocity of the particle. The velocity of the particle is less than 0, $v(t) < 0$.

(6) Expression:

The average velocity of the particle over the interval $[a, b]$.

Mathematical Translation:

$$\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{b-a}\Big[s(t)\Big]_a^b = \frac{s(b)-s(a)}{b-a}$$

(7) Expression:

The instantaneous velocity of the particle at time $t = t_1$.

Mathematical Translation:

$v(t_1) = s'(t_1)$

(8) Expression:

The acceleration of the particle at time $t = t_1$.

Mathematical Translation:

$a(t_1) = v'(t_1) = s''(t_1)$

(9) Expression:

The velocity of the particle is increasing.

Mathematical Translation:

This means that that the derivative of the velocity is positive: $a(t) = v'(t) > 0$.

(10) Expression:

The velocity of the particle is decreasing.

Mathematical Translation:

This means that that the derivative of the velocity is negative: $a(t) = v'(t) < 0$.

(11) **Expression**:

The speed of the particle

Mathematical Translation:

The speed of the particle is the absolute value of the velocity: $|v(t)|$.

(12) **Expression**:

The speed of the particle is increasing, or the particle is speeding up.

Mathematical Translation:

This means that the velocity and the acceleration have the same sign.

Therefore, either $v(t) > 0$ and $a(t) > 0$, or $v(t) < 0$ and $a(t) < 0$.

Another translation: $\frac{d}{dt}|v(t)| > 0$

(13) Expression:

The speed of the particle is decreasing, or the particle is slowing down.

Mathematical Translation:

This means that the velocity and the acceleration have different signs.
Therefore, either $v(t) > 0$ and $a(t) < 0$, or $v(t) < 0$ and $a(t) > 0$.

Another translation: $\frac{d}{dt}|v(t)| < 0$

(14) Expression:

The total distance traveled by the particle over the interval $[a, b]$.

Mathematical Translation:

$$\int_a^b |v(t)|\, dt$$

(15) Expression:

The displacement of the particle, or the net distance traveled by the particle, over the time interval $[a, b]$.

Mathematical Translation:

$$\int_a^b v(t)\, dt$$

(16) Expression:

The position of the particle at time $t = t_2$.

Mathematical Translation:

$$s(t_2) = s(t_1) + \int_{t_1}^{t_2} v(t)\, dt$$

Often $t_1 = 0$ and therefore, $s(t_1)$ is the initial position of the particle.

(17) Expression:

The particle changes direction.

Mathematical Translation:

The particle changes direction when the velocity of the particle changes from positive to negative, or from negative to positive.

(18) Expression:

The particle is farthest to the left (right).

Mathematical Translation:

This expression means we need to find the minimum (maximum) value of $s(t)$ over an interval. Therefore, we need to find the critical points of s: values of t where $s'(t) = v(t) = 0$ or $s'(t) = v(t)$ does not exist.

Consider the position of the particle at the critical points: Candidates test, First Derivative Test, Second Derivative Test.

Example 2 Particle Motion With Technology

A particle moves along a horizontal line. For $0 \leq t \leq 10$, the velocity of the particle is given by $v(t) = \dfrac{1}{1+t^3} + te^{-0.01t}\cos\left(\dfrac{t^3}{140}\right)$. The position of the particle is given by $s(t)$ and $s(0) = -2$.

(a) Find all values of t in the interval $0 \leq t \leq 10$ for which the speed of the particle is 4.

(b) Write an expression that can be used to find the position $s(t)$ of the particle at any time t. Then use this expression to find the position of the particle at time $t = 7$.

(c) Find all values of t in the interval $0 \leq t \leq 10$ at which the particle changes direction. Justify your answer.

(d) Is the speed of the particle increasing or decreasing at time $t = 5$? Give a reason for your answer.

Solution

(a) Find all values of t in the interval $0 \leq t \leq 10$ for which the speed of the particle is 4.

The speed of the particle is the absolute value of the velocity.

Solve the expression $|v(t)| = 4$ on the interval $0 \leq t \leq 10$.

(b) Write an expression that can be used to find the position $s(t)$ of the particle at any time t. Then use this expression to find the position of the particle at time $t = 7$.

The position of the particle at time t:

(c) Find all values of t in the interval $0 \le t \le 10$ at which the particle changes direction. Justify your answer.

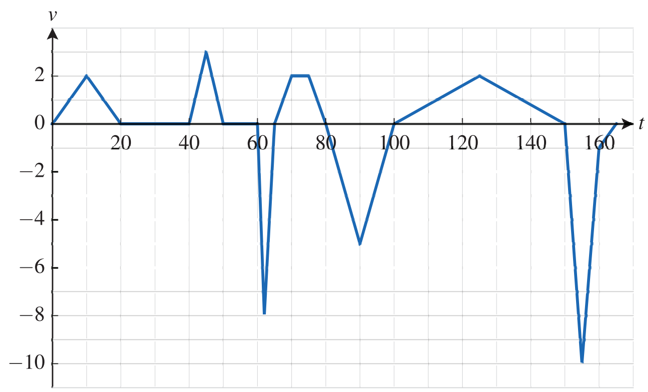
Use the graph of v to find the values of t where $v(t)$ changes sign.

(d) Is the speed of the particle increasing or decreasing at time $t = 5$? Give a reason for your answer.

Consider $v(5)$ and $a(5)$.

Example 3 Tower of Terror

The Tower of Terror is a Disney thrill ride in which guests take a service elevator to the 13th floor and experience several rapid plunges. The ride begins on the fifth floor and guests experience three random plunges. The velocity of the service elevator, for $0 \leq t \leq 165$ is modeled by a piecewise linear function shown in the figure below where t is measured in seconds and v in meters per second. Note that the ride starts on the fifth floor, $s(0) = 20$.



(a) At what times in the interval $0 \leq t \leq 165$, if any, does the elevator change direction? Give a reason for your answer.

(b) At what time in the interval $0 \leq t \leq 165$ is the elevator highest? How high is the elevator at that time?

(c) Find the total distance that the elevator travels during the time interval $0 \leq t \leq 165$.

(d) Challenge: sketch a graph of the acceleration for the elevator during the time interval $0 \leq t \leq 165$.

Solution

(a) At what times in the interval $0 \leq t \leq 165$, if any, does the elevator change direction? Give a reason for your answer.

(b) At what time in the interval $0 \leq t \leq 165$ is the elevator highest? How high is the elevator at that time?

Check the position of the elevator when the velocity is 0, and at the endpoints.

(c) Find the total distance that the elevator travels during the time interval $0 \leq t \leq 165$.

Example 4 Particle Motion, Analytically

For $0 \leq t \leq 15$, a particle moves along a horizontal line. The velocity of the particle at time t is given by $v(t) = t^2 - 12t + 27$. The particle is at position $s = -7$ at time $t = 0$.

(a) For $0 \leq t \leq 15$, when is the particle moving to the left?

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 6$.

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 10$

Solution

(a) For $0 \leq t \leq 15$, when is the particle moving to the left?

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 6$.

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = -7$

Example 5 Robot Cleaning

A (malfunctioning) Roomba vacuum moves back and forth along a straight line in a room of width 4 meters. For $0 \le t \le 60$, the velocity of the Roomba is given by a differentiable function v. Selected values of $v(t)$, where t is measured in seconds and $v(t)$ is in meters per second, are given in the table.

t (seconds)	0	10	30	50	60
$v(t)$ (meters per second)	0	0.1	−0.15	0.2	0.1

(a) Use the data in the table to estimate the value of $v'(40)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{60} |v(t)|\, dt$ in the context of the problem.

(c) Approximate the value of $\int_0^{60} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(d) There is a pet cat following the Roomba in a straight line parallel to the vacuum. The velocity of the cat is given by $c(t) = \frac{1}{4} \sin\left(\frac{\pi}{20} t\right)$, where t is measured in seconds and $c(t)$ is measured in meters per second. Find the average velocity of the cat during the interval $0 \le t \le 60$?

Solution

(a) Use the data in the table to estimate the value of $v'(40)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{60} |v(t)|\, dt$ in the context of the problem.

(c) Approximate the value of $\int_0^{60} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(d) There is a pet cat following the Roomba in a straight line parallel to the vacuum. The velocity of the cat is given by $c(t) = \frac{1}{4}\sin\left(\frac{\pi}{20}t\right)$, where t is measured in seconds and $c(t)$ is measured in meters per second. Find the average velocity of the cat during the interval $0 \le t \le 60$?

Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

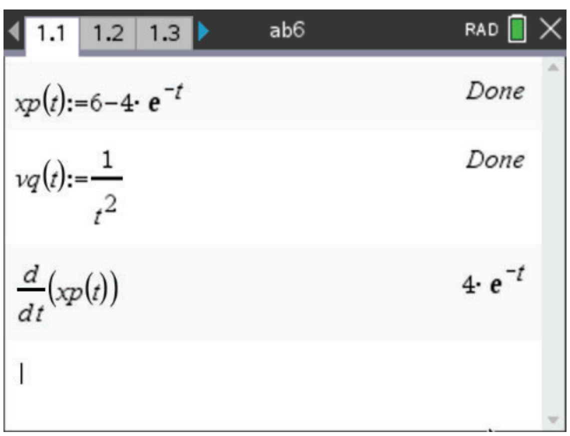
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Particle P moves along the x-axis such that, for time $t > 0,$ its position is given by $x_P(t) = 6 - 4e^{-t}.$

Particle Q moves along the y-axis such that, for time $t > 0,$ its velocity is given by $v_Q(t) = \frac{1}{t^2}.$ At time $t = 1,$ the position of particle Q is $y_Q(1) = 2.$

	Model Solution	Scoring	
(a)	Find $v_P(t),$ the velocity of particle P at time $t.$		
	$v_P(t) = x_P{}'(t) = 4e^{-t}$	Answer	**1 point**

Technology Solution



Scoring notes:

- A response that equates $x_P(t)$ with $v_P(t)$ does not earn the point.

$v_P(t) = x_P(t) = 6 - 4e^{-t}$	0
$v_P(t) = x_P(t) = 4e^{-t}$	0

- An unlabeled response earns the point.

$4e^{-t}$	1

Common Errors

(1) Linkage issues: equating position and velocity, equating velocity function with a numeric value (usually $t = 1$).

$v_P(1) = 4e^{-t}$

(2) Differentiation errors: use of the Power Rule.

$v_P(t) = -4(-t)e^{-t-1}$

(b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

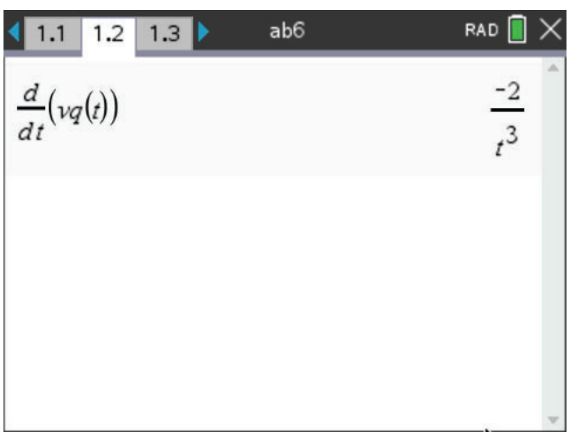
$a_Q(t) = v_Q{}'(t) = \frac{-2}{t^3}$	$a_Q(t)$	**1 point**
For $t > 0$, $a_Q(t) < 0$ and $v_Q(t) > 0$.	Considers signs of $a_Q(t)$ and $v_Q(t)$	**1 point**
Because the velocity and acceleration have opposite signs, the speed of particle Q is decreasing for all $t > 0$.	Answer with justification	**1 point**

Solution

$$v_Q(t) = \frac{1}{t^2} = t^{-2} \quad \Rightarrow \quad a_Q(t) = v'_Q(t) = -2t^{-3} = -\frac{2}{t^3}$$

For $t > 0$: $a_Q(t) = -\frac{2}{t^3} < 0$ and $v_Q(t) = \frac{1}{t^2} > 0$

Since the velocity and the acceleration have opposite signs, the speed of the particle is decreasing for all $t > 0$



Scoring notes:

- Earning the first point is not necessary for a response to be eligible to earn the second or third points; however, the response must present an expression for $a_Q(t)$ to be eligible for third point.

$a_Q(t) = \frac{2}{t^3}$ 0 - ? - ?

- A response earns the second point with either of the following statements: "$v_Q(t)$ and $a_Q(t)$ have opposite signs" or "$v_Q(t)$ and $a_Q(t)$ have the same sign." This statement, however, must be consistent with $v_Q(t)$ and the presented expression for $a_Q(t)$.

The response has considered the signs of $a_Q(t)$ and $v_Q(t)$.

- A response must earn the second point to be eligible for the third point. The answer must be consistent with the presented justification. Furthermore, responses for which $a_Q(t) > 0$ for $t > 0$ must conclude that there is no time at which the speed of the particle is decreasing.

1 - 0 - 1 not possible: must consider the signs of $v_Q(t)$ and $a_Q(t)$

Answer must be consistent:
Opposite signs: decreasing; same signs: increasing

$a_Q(t) > 0$ for $t > 0$: greater burden: must say there is no time

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- A response that indicates $v_Q(t) < 0$ does not earn the third point, even if the answer and justification are consistent with a reported sign of $a_Q(t)$.

Can earn the second point for considering the signs of $v_Q(t)$ and $a_Q(t)$

$v_Q(t) = \frac{1}{t^2} < 0$ for $t > 0$ and $a_Q(t) < 0$

Common Errors

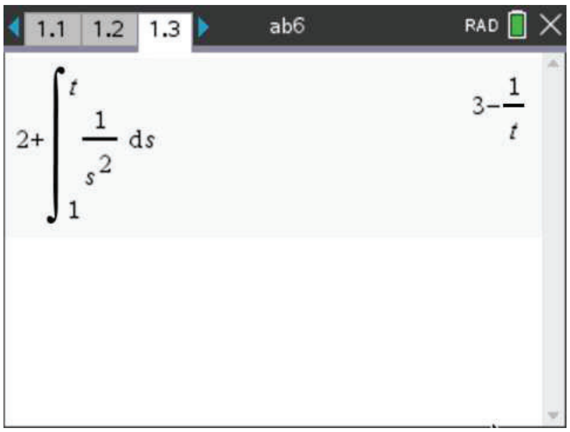
(1) Many responses did not consider the sign of the velocity function.

(2) Particle is indeed slowing when the velocity is positive and the acceleration is negative.

(c) Find $y_Q(t)$, the position of particle Q at time t.

$y_Q(t) = y_Q(1) + \int_1^t \frac{1}{s^2}\,ds$	Integral	**1 point**
	Uses initial condition	**1 point**
$= 2 - \left(\frac{1}{s}\Big\vert_1^t\right) = 2 - \frac{1}{t} + 1 = 3 - \frac{1}{t}$	Answer	**1 point**

Technology Solution



Scoring notes:

- A response that presents $\int_1^t \frac{1}{t^2}\,dt$ (using the same variable as a limit and integrand function) does not earn the first point unless it is followed by an attempt at integration.

Use of the same variable as a limit and in the integrand function
(Chief Reader decision)

- A response that presents either $\int \frac{1}{t^2}\,dt$ or $-\frac{1}{t}$ (with no integral) earns the first point. If the response continues and presents $2 = -1 + C,$ then the response earns the second point.

$\int \frac{1}{t^2}\,dt = -\frac{1}{t} + C$ (on this solution trail)

$$y_Q(1) = 2 = -\frac{1}{1} + C \quad \Rightarrow \quad C = 3 \quad \Rightarrow \quad y_Q(t) = 3 - \frac{1}{t}$$

$\int \frac{1}{t^2}\,dt$ (on the same solution trail?)

- A response that presents only $y_Q(t) = -\frac{1}{t} + 3$ will earn all 3 points. Note that the right side of this equation suffices to earn all points. A response of $y_Q(t) = -\frac{1}{t} + C,$ where $C \neq 3,$ (with no additional supporting work) earns only the first point.

$y_Q(t) = -\frac{1}{t} + 3$	1 - 1 - 1
$y_Q(t) = -\frac{1}{t} + C$	1 - 0 - 0

Common Errors

(1) Most responses used an indefinite integral.

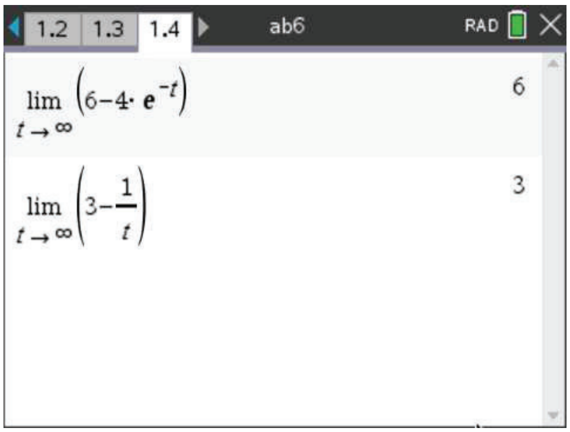
(2) Same variable as a limit and as the variable of integration.

(3) Some errors in finding an antiderivative. (1 - 1 - 0)

(d) As $t \to \infty$, which particle will eventually be farther from the origin? Give a reason for your answer.

For particle P, $\lim_{t\to\infty}\left(6 - 4e^{-t}\right) = 6$. For particle Q, $\lim_{t\to\infty}\left(3 - \frac{1}{t}\right) = 3$.	One correct limit	**1 point**
Because $6 > 3$, particle P will eventually be farther from the origin.	Answer with reason	**1 point**

Technology Solution



Scoring notes:

- A response with an incorrect $y_Q(t)$ from part (c) is eligible for both points in part (d) provided $y_Q(t)$ is a non-constant function. The second point is earned for a consistent answer with reason, and limits correct for particle P and the presented $y_Q(t)$.

$y_Q(t)$ incorrect, non-constant: read with the student.

- Responses that present statements such as "$6 - 4e^{-t}$ approaches 6" or "Q goes to 3" earn the first point and are eligible for the second point.

As $t \to \infty$, $Q \to 3$

- A response that treats ∞ as an input for $x_P(t)$ or $y_Q(t)$, such as "$6 - 4e^{-\infty}$" or "$3 - \frac{1}{\infty}$" is not eligible for the second point.

Use of ∞ as a number.

Common Errors

(1) Poor communication: use of infinity as a number.

(2) Some responses wrote about horizontal asymptotes.
But response needed to be more specific.

Example 6 Particle Motion With Technology, Revisited

A particle moves along a horizontal line. For $0 \le t \le 10$, the velocity of the particle is given by $v(t) = \dfrac{1}{1+t^3} + te^{-0.01t}\cos\left(\dfrac{t^3}{140}\right)$. The position of the particle is given by $s(t)$ and $s(0) = -2$.

(a) Find all values of t in the interval $0 \le t \le 10$ for which the speed of the particle is 4.

(b) Write an expression that can be used to find the position $s(t)$ of the particle at any time t. Then use this expression to find the position of the particle at time $t = 7$.

(c) Find all values of t in the interval $0 \le t \le 10$ at which the particle changes direction. Justify your answer.

(d) Is the speed of the particle increasing or decreasing at time $t = 5$? Give a reason for your answer.